THE SLOW STATIONARY FLOW OF A VISCOUS FLUID ABOUT A POROUS SPHERE

(MEDLENNOE STATSIONARNOE OBTEKANIE PORISTOI SFERY VIAZKOI ZHIDKOST'IU)

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A series of papers [1-9] have been devoted to the flow of an incompressible viscous fluid in a space bounded by a porous wall. In these papers the permeability velocity on the porous wall is assumed to be given. However, problems do exist in which the permeability velocity is not known in advance and is determined in the process of solution.

The properties of the porous medium of the body which is subjected to the flow are important for such problems. In the case of the flow of a viscous fluid about a porous shell, whose thickness is small in comparison to the minimum defining dimension of the region of flow, the solution can be built without approximation from filtration theory. For the boundary conditions on the porous surface, the component of the permeability velocity which lies in the tangent plane to the given point of the porous surface is taken as equal to zero and the normal component of the velocity vector is taken to be continuous across the porous boundary.

It should be noted further that another approach to problems with permeable boundaries is also possible when flow in a singly-connected region is under consideration. For example, in [10] the permeability velocity was assumed to be proportional to the difference between the fluid pressure and the pressure in the porous medium. Problems of the flow of a stationary stream of idealized fluid about a porous circular cylinder and a closed porous shell have been considered in [11-14] and in dissertations*.

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* Baichorov, Kh.Ia., The flow of a stream of an idealized incompressible fluid about some porous obstacles. Dissertation, MGU, 1949. Kolosovskaia, A.K., Some planar problems of the motion of a permeable body in an idealized incompressible fluid. Dissertation, MGU, 1953.

Let a porous sphere of radius a and thickness $\delta \ll a$ be flowed about by a stationary axisymmetric stream of an incompressible viscous fluid. We note at once that axial symmetry is possible only by virtue of homogeneous porosity. In the spherical coordinates r, φ, θ we have because of symmetry

$$v_\varphi \equiv 0, \ \partial v_\theta/\partial\varphi \equiv 0, \ \partial v_r/\partial\varphi \equiv 0$$

At infinity we have the conditions

$$v_r = -U\cos\theta, \qquad v_\theta = U\sin\theta, \qquad r\to\infty \tag{1}$$

Here U is the flow velocity at infinity. We shall solve the problem in Stokesian approximation. As is well known [15], the stream function $\psi(r, \theta)$ satisfies the equation

$$DD\psi = 0, \qquad D = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) (D \text{ is Stokes' operator}) \tag{2}$$

The projections of the velocity vector are represented by the equalities

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{3}$$

and the pressure is determined from the equations (μ is the viscosity)

$$\frac{\partial p}{\partial r} = \frac{\mu}{r^2\sin\theta}\frac{\partial D\psi}{\partial\theta}, \qquad \frac{\partial p}{\partial\theta} = -\frac{\mu}{\sin\theta}\frac{\partial D\psi}{\partial r} \tag{4}$$

The stream function, defined for all space, has the form

$$\psi = \sin^2\theta\,(Ar^4 + Br + Cr^2 + E/r) \qquad (A, B, C, E = \text{const}) \tag{5}$$

From the continuity equation and the no-slip condition there follows

$$\left(\frac{\partial v_r}{\partial r} + \frac{2v_r}{r}\right)_{r=a} = 0, \qquad v_\theta\,|_{r=a} = 0 \tag{6}$$

The continuity equation of the normal velocity component across the porous boundary has the form

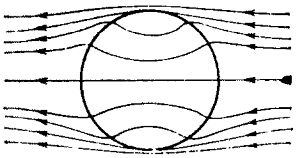
$$v_r\,|_{r=a+0} = v_r\,|_{r=a-0} \tag{7}$$

We shall consider the flow exterior to a porous sphere, i.e. for $r \geqslant a$. By virtue of (3) and (5) we have

$$v_r = 2\cos\theta\left(A_+r^3 + \frac{B_+}{r} + C_+ + \frac{E_+}{r^3}\right)$$

$$v_\theta = -\sin\theta\left(4A_+r^2 + \frac{B_+}{r} + 2C_+ - \frac{E_+}{r^3}\right)$$

Here values of the constants for the exterior of the sphere are indicated by a plus index. From (6) we obtain



$$4A_+a^2 + \frac{B_+}{a} + 2C_+ - \frac{E_+}{a^3} = 0 \tag{8}$$

Equation (8) and the condition (1) will be satisfied if we set (s is a parameter)

$$A_+ = 0, \qquad B_+ = -aU(s-1), \qquad C_+ = -{}^1/_2U, \qquad E_+ = -sUa^3 \tag{9}$$

We shall now consider the flow within the porous sphere, i.e. for $r \leqslant a$. Applying conditions (6) and (7) in exactly the same way and also the condition of regularity of the flow in the interior region (the absence of sources and sinks), we obtain

$$A_- = {}^1/_2U(4s-1), \qquad B_- = E_- = 0, \qquad C_- = -U(4s-1) \tag{10}$$

The values of the constants for $r \leqslant a$ are indicated by the minus index. Substituting the values of the constants in (3) and (5),,we obtain

$$\psi = \begin{cases} -U\sin^2\theta\left[\dfrac{r^2}{2} + ar(s-1) + s\dfrac{a^3}{r}\right] & (r \geqslant a) \\ -U\sin^2\theta\,(4s-1)\left(r^2 - \dfrac{r^4}{2a^2}\right) & (r \leqslant a) \end{cases}$$

$$v_r = \begin{cases} -2U\cos\theta\left[\dfrac{1}{2} + \dfrac{a}{r}(s-1) + s\left(\dfrac{a}{r}\right)^3\right] & (r \geqslant a) \\ -2U\cos\theta\,(4s-1)\left(1 - \dfrac{r^2}{2a^2}\right) & (r \leqslant a) \end{cases} \tag{11}$$

$$v_\theta = \begin{cases} U\sin\theta\left[1 + \dfrac{a}{r}(s-1) - s\left(\dfrac{a}{r}\right)^3\right] & (r \geqslant a) \\ 2U\sin\theta\,(4s-1)\left(1 - \dfrac{r^2}{a^2}\right) & (r \leqslant a) \end{cases}$$

From Formulas (11) it is seen that the projections of the velocity are continuous and the stream function maintains a continuous derivative across the porous surface.

Formulas (11) define a single-parameter family of regular flows with a normally permeable boundary. For $s = 1/4$ we obtain the solution to the Stokes problem of the flow about a sphere with a permeable boundary. In this case, as follows from (11), the velocity within the sphere is equal to zero, which corresponds fully with the physical sense of the problem. From (11) it is possible to determine the permeability velocity

$$v_0 = v_r\,|_{r=a} = -U(4s-1)\cos\theta \tag{12}$$

For $s > 1/4$ (the case of permeable flows) the streamlines have the typical form shown in the figure. From Equations (4) it is possible to obtain the pressure distribution

$$p(r,\theta)=\begin{cases} p_0-2\mu U(s-1)\,ar^{-2}\cos\theta & (r>a)\\ p_0+10\mu U(4s-1)\,ra^{-2}\cos\theta & (r<a)\end{cases} \tag{13}$$

where p_0 is an arbitrary constant. From (13) it is seen that the pressure is discontinuous on the porous surface

$$\Delta p_a=p\,|_{r=a+0}-p\,|_{r=a-0}=6\mu U\,\frac{2-7s}{a}\cos\theta \tag{14}$$

The range of variation of the parameter s for physically possible flows is determined from the obvious conditions

$$v_0<0,\quad \Delta p_a>0\quad \text{for}\quad 0<\theta<\pi/2,\qquad v_0>0,\quad \Delta p_a<0\quad \text{for}\quad \pi/2<\theta<\pi$$

From (12) and (14) it then follows that

$$1/4\leqslant s<2/7 \tag{15}$$

Comparing (12) and (14), we obtain

$$\Delta p_a=-6\mu\,\frac{v_0}{a}\,\frac{2-7s}{4s-1} \tag{16}$$

We shall now determine the resultant resistance of a porous sphere. Because of the axial symmetry it will be directed along the axis of symmetry of the flow, the z-axis; we have

$$R_z=2\pi a^2\int_0^{\pi}[p_{rr}\cos\theta-p_{r\theta}\sin\theta]_a\sin\theta\,d\theta \tag{17}$$

Here in the square brackets the difference of the maximum values are taken for $r = a + 0$ and $r = a - 0$. Expressing the values of p_{rr} and $p_{r\theta}$ for $r = a$ by p, v_r, v_θ according to Newton's law and then using Formulas (11) and (13), the integrand of (17) can be easily calculated. Integrating, we have

$$R=-{}^8/_3\pi a\mu U\,(11-35s) \tag{18}$$

We obtain the Stokes formula from (18) with $s = 1/4$.

We shall consider the inverse problem. A porous sphere moves in a viscous liquid with constant velocity U in the direction of the z-axis. Let ψ^* be the stream function and v_r^*, v_θ^* the velocity components of the inverted motion. We shall obtain the solution, letting

$$\psi^*=\psi+{}^1/_2Ur^2\sin^2\theta,\qquad v_r^*=v_r+U\cos\theta,\qquad v_\theta^*=v_\theta-U\sin\theta \tag{19}$$

The pressure distribution in this case will be just the same as in the problem of flow about the body, i.e. it will be determined by Formula (13). The proof follows from (4), (19) and from the fact that $D(r^2 \sin^2 \theta) = 0$; the resultant resistance of the porous sphere in the given case will also be determined by Expression (19).

The parameter s must be determined from experiment. It is easy to determine if the coefficient of permeability of the porous material of which the shell is constructed is known. Actually, Expression (16) can be written in the following form:

$$v_0 = -\frac{k}{\mu}\frac{\Delta p_a}{\delta}, \qquad k = \frac{a\delta}{6}\frac{4s-1}{2-7s} \tag{20}$$

which is the Darcy law; the constant k is the coefficient of permeability.

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